CHAPTER 4

LIMITS AND DIFFERENTIATION

LEARNING OBJECTIVES

Upon completion of this chapter, you should be able to do the following:

1. Define a limit, find the limit of indeterminate forms, and apply limit formulas.
2. Define an infinitesimal, determine the sum and product of infinitesimals, and restate the concept of infinitesimals.
3. Identify discontinuities in a function.
4. Relate increments to differentiation, apply the general formula for differentiation, and find the derivative of a function using the general formula.

INTRODUCTION

Limits and differentiation are the beginning of the study of calculus, which is an important and powerful method of computation.

LIMIT CONCEPT

The study of the limit concept is very important, for it is the very heart of the theory and operation of calculus. We will include in this section the definition of limit, some of the indeterminate forms of limits, and some limit formulas, along with example problems.

DEFINITION OF LIMIT

Before we start differentiation, we must understand certain concepts. One of these concepts deals with the limit of a function. Many times you will need to find the value of the limit of a function.

The discussion of limits will begin with an intuitive point of view.

We will work with the equation

$$y = f(x) = x^2$$

which is shown in figure 4-1. Point P represents the point corresponding to

$$y = 16$$

and

$$x = 4$$

The behavior of y for given values of x near the point

$$x = 4$$

is the center of the discussion. For the present we will exclude point P, which is encircled on the graph.

We will start with values lying between and including

$$x = 2$$

and

$$x = 6$$

indicated by interval AB in figure 4-1, view A. This interval may be written as

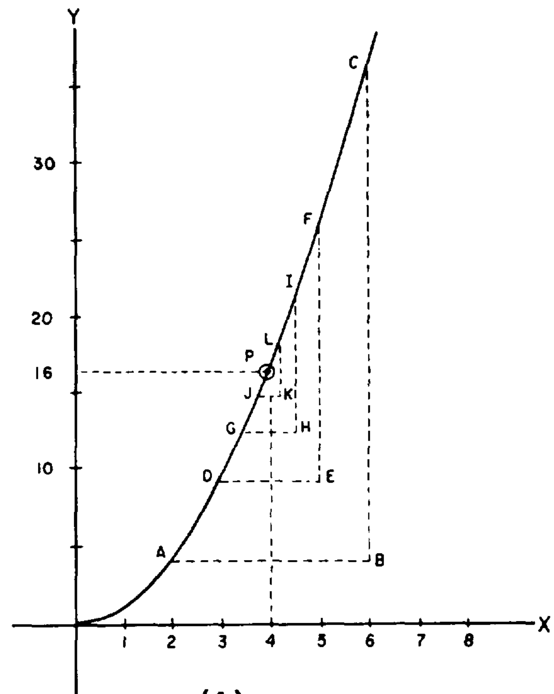
$$2 \leq x \leq 6 \quad \text{or} \quad 0 \leq |x - 4| \leq 2$$

The corresponding interval for y is between and includes

$$y = 4$$

and

$$y = 36$$



(A)

Interval of x		Interval of $f(x)$	
2 - 6	AB	4 - 36	BC
3 - 5	DE	9 - 25	EF
3.5 - 4.5	GH	12.25 - 20.25	HI
3.9 - 4.1	JK	15.21 - 16.81	KL

(B)

Figure 4-1.—(A) Graph of $y = x^2$; (B) value chart.

We now take a smaller interval, DE, about $x = 4$ by using values of

$$x = 3$$

and

$$x = 5$$

and find the corresponding interval for y to be between

$$y = 9$$

and

$$y = 25$$

inclusively.

These intervals for x and y are written as

$$0 \leq |x - 4| \leq 1$$

and

$$9 \leq y \leq 25$$

As we diminish the interval of x around

$$x = 4 \text{ (intervals } GH \text{ and } JK\text{)}$$

we find the values of

$$y = x^2$$

to be grouped more and more closely around

$$y = 16$$

This is shown by the chart in figure 4-1, view B.

Although we have used only a few intervals of x in the discussion, you should easily see that we can make the values about y group as closely as we desire by merely limiting the values assigned to x about

$$x = 4$$

Because the foregoing is true, we may now say that the limit of x^2, as x approaches 4, results in the value 16 for y, and we write

$$\lim_{x \to 4} x^2 = 16$$

In the general form we may write

$$\lim_{x \to a} f(x) = L \tag{4.1}$$

Equation (4.1) means that as x approaches a, the limit of $f(x)$ will approach L, where L is the limit of $f(x)$ as x approaches a. No statement is made about $f(a)$, for it may or may not exist, although the limit of $f(x)$, as x approaches a, is defined.

We are now ready to define a limit.

Let $f(x)$ be defined for all x in the interval near

$$x = a$$

but not necessarily at

$$x = a$$

Then there exists a number, L, such that for every positive number ε (epsilon), however small,

$$|f(x) - L| < \varepsilon$$

provided that we may find a positive number δ (delta) such that

$$0 < |x - a| < \delta$$

Then we say L is the limit of $f(x)$ as x approaches a, and we write

$$\lim_{x \to a} f(x) = L$$

This means that for every given number $\varepsilon > 0$, we must find a number δ such that the difference between $f(x)$ and L is smaller than the number ε whenever

$$0 < |x - a| < \delta$$

EXAMPLE: Suppose we are given $\varepsilon = 0.1$ and

$$\lim_{x \to 1} \frac{x^2 + x - 2}{3(x - 1)} = 1$$

find a $\delta > 0$.

SOLUTION: We must find a number δ such that for all points except

$$x = 1$$

we have the difference between $f(x)$ and 1 smaller than 0.1.

We write

$$\left| \frac{x^2 + x - 2}{3(x - 1)} - 1 \right| < 0.1$$

and

$$\frac{x^2 + x - 2}{3(x - 1)} - 1$$

$$= \frac{(x + 2)(x - 1)}{3(x - 1)} - 1$$

and we consider only values where

$$x \neq 1$$

Simplifying the first term, we have

$$\frac{(x + 2)(x - 1)}{3(x - 1)} = \frac{x + 2}{3}$$

Finally, combine terms as follows:

$$\frac{x + 2}{3} - 1 = \frac{x + 2 - 3}{3} = \frac{x - 1}{3}$$

so that

$$\left| \frac{x-1}{3} \right| < 0.1$$

or

$$| x - 1 | < 0.3$$

Therefore, $\delta = 0.3$ and we have fulfilled the definition of the limit.

If the limit of a function exists, then

$$\lim_{x \to a} f(x) = f(a)$$

So we can often evaluate the limit by substitution.

For instance, to find the limit of the function $x^2 - 3x + 2$ as x approaches 3, we substitute 3 for x in the function. Then

$$f(3) = 3^2 - 3(3) + 2$$

$$= 9 - 9 + 2$$

$$= 2$$

Since x is a variable, it may assume a value as close to 3 as we wish; and the closer we choose the value of x to 3, the closer $f(x)$ will approach the value of 2. Therefore, 2 is called the limit of $f(x)$ as x approaches 3, and we write

$$\lim_{x \to 3} (x^2 - 3x + 2) = 2$$

PRACTICE PROBLEMS:

Find the limit of each of the following functions:

1. $\lim_{x \to 1} \frac{2x^2 - 1}{2x - 1}$

2. $\lim_{x \to 2} (x^2 - 2x + 3)$

3. $\lim_{x \to a} \frac{x^2 - a}{a}$

4. $\lim_{t \to 0} (5t^2 - 3t + 2)$

5. $\lim_{E \to 6} \frac{E^3 - E}{E - 1}$

6. $\lim_{Z \to 0} \frac{Z^2 - 3Z + 2}{Z - 4}$

ANSWERS:

1. 1
2. 3
3. $a - 1$
4. 2
5. 42
6. $\frac{-1}{2}$

INDETERMINATE FORMS

When the value of a limit is obtained by substitution and it assumes any of the following forms, another method for finding the limit must be used:

$$\frac{0}{0}, \frac{\infty}{\infty}, (\infty)0, 0^0, \infty^0, 1^\infty$$

These are called *indeterminate forms.*

There are many methods of evaluating indeterminate forms. *Two methods of evaluating indeterminate forms* *are (1) factoring and (2) division of the numerator and denominator by powers of the variable.*

Sometimes factoring will resolve an indeterminate form.

EXAMPLE: Find the limit of

$$\frac{x^2 - 9}{x - 3} \text{ as } x \text{ approaches } 3$$

SOLUTION: By substitution we find

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = \frac{0}{0}$$

which is an indeterminate form and is therefore excluded as a possible limit. We must now search for a method to find the limit. Factoring is attempted, which results in

$$\frac{x^2 - 9}{x - 3} = \frac{(x + 3)(x - 3)}{x - 3}$$

$$= x + 3$$

so that

$$\lim_{x \to 3} (x + 3) = 6$$

and we have a determinate limit of 6.

Another indeterminate form is often met when we try to find the limit of a function as the independent variable approaches infinity.

EXAMPLE: Find the limit of

$$\frac{x^4 + 2x^3 - 3x^2 + 2x}{3x^4 - 2x^2 + 1}$$

as $x \to \infty$.

SOLUTION: If we let x approach infinity in the original expression, the result will be

$$\lim_{x \to \infty} \frac{x^4 + 2x^3 - 3x^2 + 2x}{3x^4 - 2x^2 + 1} = \frac{\infty}{\infty}$$

which must be excluded as an indeterminate form. However, if we divide both numerator and denominator by x^4, we obtain

$$\lim_{x\to\infty} \frac{1 + \frac{2}{x} - \frac{3}{x^2} + \frac{2}{x^3}}{3 - \frac{2}{x^2} + \frac{1}{x^4}}$$

$$= \frac{1 + 0 - 0 + 0}{3 - 0 + 0}$$

$$= \frac{1}{3}$$

and we have a determinate limit of $\frac{1}{3}$.

PRACTICE PROBLEMS:

Find the limit of the following:

1. $\lim\limits_{x\to 2} \dfrac{x^2 - 4}{x - 2}$

2. $\lim\limits_{x\to\infty} \dfrac{2x + 3}{7x - 6}$

3. $\lim\limits_{a\to 0} \dfrac{2a^2b - 3ab^2 + 2ab}{5ab - a^3b^2}$

4. $\lim\limits_{x\to 3} \dfrac{x^2 - x - 6}{x - 3}$

5. $\lim\limits_{x\to a} \dfrac{x^4 - a^4}{x - a}$

6. $\lim\limits_{a\to 0} \dfrac{(x - a)^2 - x^2}{a}$

ANSWERS:

1. 4

2. $\frac{2}{7}$

3. $\frac{2-3b}{5}$

4. 5

5. $4a^3$

6. $-2x$

LIMIT THEOREMS

To obtain results in calculus, we will frequently operate with limits. The proofs of theorems shown in this section will be omitted in the interest of brevity. The theorems will be stated and examples will be given.

Assume that we have three simple functions of x. Further, let these functions [$f(x)$, $g(x)$, and $h(x)$] have separate limits such that

$$\lim_{x \to a} f(x) = A$$

$$\lim_{x \to a} g(x) = B$$

$$\lim_{x \to a} h(x) = C$$

Theorem 1. *The limit of the sum of two functions is equal to the sum of the limits:*

$$\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

$$= A + B$$

This theorem may be extended to include any number of functions, such as

$$\lim_{x \to a} [f(x) + g(x) + h(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) + \lim_{x \to a} h(x)$$

$$= A + B + C$$

EXAMPLE: Find the limit of

$$(x - 3)^2 \text{ as } x \to 3$$

SOLUTION:

$$\lim_{x \to 3} (x - 3)^2 = \lim_{x \to 3} (x^2 - 6x + 9)$$

$$= \lim_{x \to 3} x^2 - \lim_{x \to 3} 6x + \lim_{x \to 3} 9$$

$$= 9 - 18 + 9$$

$$= 0$$

Theorem 2. *The limit of a constant, c, times a function, f(x), is equal to the constant, c, times the limit of the function:*

$$\lim_{x \to a} cf(x) = c \lim_{x \to a} f(x) = cA$$

EXAMPLE: Find the limit of

$$2x^2 \text{ as } x \to 3$$

SOLUTION:

$$\lim_{x \to 3} 2x^2 = 2 \lim_{x \to 3} x^2$$

$$= (2)(9)$$

$$= 18$$

Theorem 3. *The limit of the product of two functions is equal to the product of their limits*:

$$\lim_{x \to a} f(x)g(x) = \left[\lim_{x \to a} f(x)\right]\left[\lim_{x \to a} g(x)\right] = AB$$

EXAMPLE: Find the limit of

$$(x^2 - x)(\sqrt{2x}) \text{ as } x \to 2$$

SOLUTION:

$$\lim_{x \to 2} (x^2 - x)(\sqrt{2x}) = \left[\lim_{x \to 2} (x^2 - x)\right]\left[\lim_{x \to 2} \sqrt{2x}\right]$$

$$= (4 - 2)(\sqrt{4})$$

$$= 4$$

Theorem 4. *The limit of the quotient of two functions is equal to the quotient of their limits, provided the limit of the divisor is not equal to zero:*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{A}{B}, \text{ if } B \neq 0$$

EXAMPLE: Find the limit of

$$\frac{3x^2 + x - 6}{2x - 5} \text{ as } x \to 3$$

SOLUTION:

$$\lim_{x \to 3} \frac{3x^2 + x - 6}{2x - 5}$$

$$= \frac{\lim_{x \to 3} 3x^2 + x - 6}{\lim_{x \to 3} 2x - 5}$$

$$= 24$$

PRACTICE PROBLEMS

Find the limits of the following, using the theorem indicated:

1. $x^2 + x + 2$ as $x \to 1$ (Theorem 1)
2. $7(x^2 - 13)$ as $x \to 4$ (Theorem 2)
3. $(5x^4)(x - 1)$ as $x \to 2$ (Theorem 3)
4. $\dfrac{2x^2 + x - 4}{3x - 7}$ as $x \to 3$ (Theorem 4)

ANSWERS:

1. 4
2. 21
3. 80
4. $\frac{17}{2}$

INFINITESIMALS

In chapter 3, we found the slope of a curve at a given point by taking very small increments of Δy and Δx, and the slope was said to be equal to $\frac{\Delta y}{\Delta x}$. This section will be a continuation of this concept.

DEFINITION

A variable that approaches 0 as a limit is called an *infinitesimal.* This may be written as

$$\lim V = 0$$

or

$$V \to 0$$

and means, as recalled from a previous section of this chapter, that the numerical value of V becomes and remains less than any positive number ε.

If the

$$\lim V = L$$

then

$$\lim V - L = 0$$

which indicates the difference between a variable and its limit is an *infinitesimal*. Conversely, if the difference between a variable and a constant is an infinitesimal, then the variable approaches the constant as a limit.

EXAMPLE: As x becomes increasingly large, is the term $\frac{1}{x^2}$ an infinitesimal?

SOLUTION: By the definition of infinitesimal, if $\frac{1}{x^2}$ approaches 0 as x increases in value, then $\frac{1}{x^2}$ is an infinitesimal. We see that $\frac{1}{x^2} \rightarrow 0$ and is therefore an infinitesimal.

EXAMPLE: As x approaches 2, is the expression $\frac{x^2 - 4}{x - 2} - 4$ an infinitesimal?

SOLUTION: By the converse of the definition of infinitesimal, if the difference between $\frac{x^2 - 4}{x - 2}$ and 4 approaches 0, as x approaches 2, the expression $\frac{x^2 - 4}{x - 2} - 4$ is an infinitesimal. By direct substitution we find an indeterminate form; therefore, we make use of our knowledge of indeterminates and write

$$\frac{x^2 - 4}{x - 2} = \frac{(x + 2)(x - 2)}{x - 2} = x + 2$$

and

$$\lim_{x \rightarrow 2} (x + 2) = 4$$

The difference between 4 and 4 is 0, so the expression $\frac{x^2 - 4}{x - 2} - 4$ is an infinitesimal as x approaches 2.

SUMS

An infinitesimal is a variable that approaches 0 as a limit. We state that ε and δ, in figure 4-2, are infinitesimals because they both approach 0 as shown.

Theorem 1. *The algebraic sum of any number of infinitesimals is an infinitesimal.*

In figure 4-2, as ε and δ approach 0, notice that their sum approaches 0; by definition this sum is an infinitesimal. This approach may be used for the sum of any number of infinitesimals.

$\delta \backslash \varepsilon$	1	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{64}$	$\frac{1}{256}$	$\to 0$
1	2	$\frac{5}{4}$	$\frac{17}{16}$	$\frac{65}{64}$	$\frac{257}{256}$	
$\frac{1}{4}$	$\frac{5}{4}$	$\frac{1}{2}$	$\frac{5}{16}$	$\frac{17}{64}$	$\frac{65}{256}$	
$\frac{1}{16}$	$\frac{17}{16}$	$\frac{5}{16}$	$\frac{1}{8}$	$\frac{5}{64}$	$\frac{17}{256}$	
$\frac{1}{64}$	$\frac{65}{64}$	$\frac{17}{64}$	$\frac{5}{64}$	$\frac{1}{32}$	$\frac{5}{256}$	
$\frac{1}{256}$	$\frac{257}{256}$	$\frac{65}{256}$	$\frac{17}{256}$	$\frac{5}{256}$	$\frac{1}{128}$	
$\downarrow 0$						$\searrow 0$

Figure 4-2.—Sums of infinitesimals.

PRODUCTS

Theorem 2. *The product of any number of infinitesimals is an infinitesimal.*

In figure 4-3, the product of two infinitesimals, ε and δ, is an infinitesimal as shown. The product of any number of infinitesimals is also an infinitesimal by the same approach as shown for two numbers.

Theorem 3. *The product of a constant and an infinitesimal is an infinitesimal.*

This may be shown, in figure 4-3, by holding either ε or δ constant and noticing their product as the variable approaches 0.

$\delta \backslash \varepsilon$	1	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{64}$	$\frac{1}{256}$	$\to 0$
1	1	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{64}$	$\frac{1}{256}$	
$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{64}$	$\frac{1}{256}$	$\frac{1}{1024}$	
$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{64}$	$\frac{1}{256}$	$\frac{1}{1024}$	$\frac{1}{4096}$	
$\frac{1}{64}$	$\frac{1}{64}$	$\frac{1}{256}$	$\frac{1}{1024}$	$\frac{1}{4096}$	$\frac{1}{16384}$	
$\frac{1}{256}$	$\frac{1}{256}$	$\frac{1}{1024}$	$\frac{1}{4096}$	$\frac{1}{16384}$	$\frac{1}{65536}$	
$\downarrow 0$						$\searrow 0$

Figure 4-3.—Products of infinitesimals.

CONCLUSIONS

The term infinitesimal was used to describe the term Δx as it approaches zero. The quantity Δx was called an increment of x, where an increment was used to imply that we made a change in x. Thus $x + \Delta x$ indicates that we are holding x constant and changing x by a variable amount which we will call Δx.

A very small increment is sometimes called a *differential*. A small Δx is indicated by dx. The differential of θ is $d\theta$ and that of y is dy. The limit of Δx as it approaches zero is, of course, zero; but that does not mean the ratio of two infinitesimals cannot be a real number or a real function of x. For instance, no matter how small Δx is chosen, the ratio $\frac{dx}{dx}$ will still be equal to 1.

In the section on indeterminate forms, a method for evaluating the form $\frac{0}{0}$ was shown. This form results whenever the limit takes the form of one infinitesimal over another. In every case the limit was a real number.

DISCONTINUITIES

The discussion of discontinuties will be based on a comparison to continuity.

A function, $f(x)$, is *continuous* at $x = a$ if the following three conditions are met:

1. *$f(x)$ is defined at $x = a$.*
2. *The limit of $f(x)$ exists as x approaches a or $x \to a$.*
3. *The value of $f(x)$ at $x = a$ is equal to the limit of $f(x)$ at $x = a$ or $\lim_{x \to a} f(x) = f(a)$.*

If a function $f(x)$ is not continuous at

$$x = a$$

then it is said to be *discontinuous* at

$$x = a$$

We will use examples to show the above statements.

EXAMPLE: In figure 4-4, is the function

$$f(x) = x^2 + x - 4$$

continuous at $f(2)$?

SOLUTION:

$$f(2) = 4 + 2 - 4$$

$$= 2$$

and

$$\lim_{x \to 2} x^2 + x - 4 = 2$$

and

$$\lim_{x \to 2} f(x) = f(2)$$

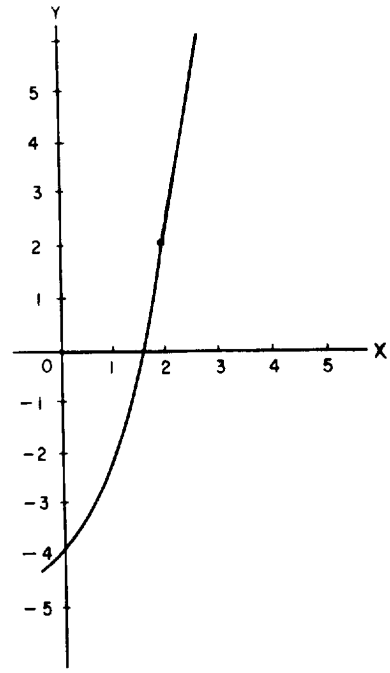


Figure 4-4.—Function $f(x) = x^2 + x - 4$.

Therefore, the curve is continuous at

$$x = 2$$

EXAMPLE: In figure 4-5, is the function

$$f(x) = \frac{x^2 - 4}{x - 2}$$

continuous at $f(2)$?

SOLUTION:

$f(2)$ is undefined at

$$x = 2$$

and the function is therefore discontinuous at

$$x = 2$$

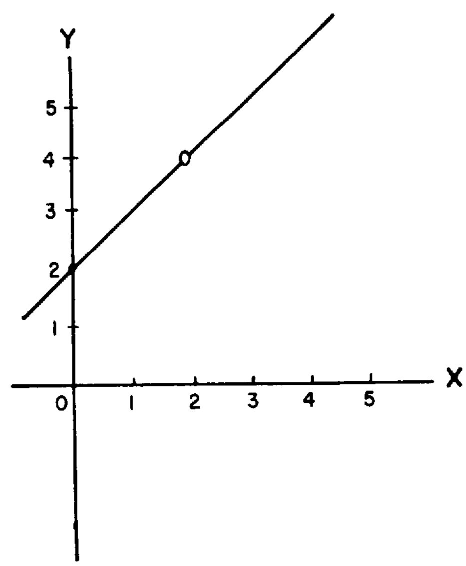


Figure 4-5.—Function $f(x) = \dfrac{x^2 - 4}{x - 2}$.

However, by extending the original equation of $f(x)$ to read

$$f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2}, x \neq 2 \\ \\ 4, x = 2 \end{cases}$$

we will have a continuous function at

$$x = 2$$

NOTE: The value of 4 at $x = 2$ was found by factoring the numerator of $f(x)$ and then simplifying.

A common kind of discontinuity occurs when we are dealing with the tangent function of an angle. Figure 4-6 is the graph of the tangent as the angle varies from 0° to 90°; that is, from 0 to $\frac{\pi}{2}$. The value of the tangent at $\frac{\pi}{2}$ is undefined. Thus the function is said to be discontinuous at $\frac{\pi}{2}$.

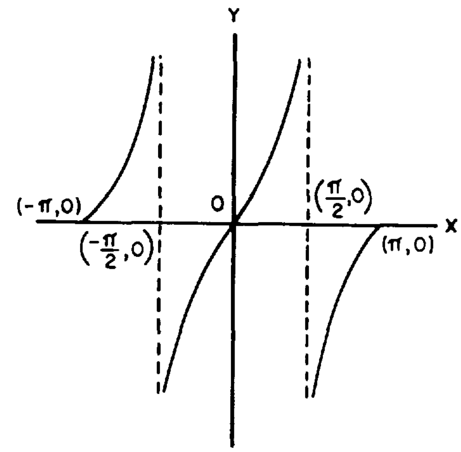


Figure 4-6.—Graph of tangent function.

PRACTICE PROBLEMS:

In the following definitions of the functions, find where the functions are discontinuous and then extend the definitions so that the functions are continuous:

1. $f(x) = \dfrac{x^2 - x - 2}{x - 2}$

2. $f(x) = \dfrac{x^2 + 2x - 3}{x + 3}$

3. $f(x) = \dfrac{x^2 + x - 12}{3x - 9}$

ANSWERS:

1. $x = 2, f(2) = 3$
2. $x = -3, f(-3) = -4$
3. $x = 3, f(3) = \frac{7}{3}$

INCREMENTS AND DIFFERENTIATION

In this section we will extend our discussion of limits and examine the idea of the derivative, the basis of differential calculus.

We will assume we have a particular function of x, such that

$$y = x^2$$

If x is assigned the value 10, the corresponding value of y will be $(10)^2$ or 100. Now, if we increase the value of x by 2, making it 12, we may call this increase of 2 an increment or Δx. This results in an increase in the value of y, and we may call this increase an increment or Δy. From this we write

$$\begin{aligned} y + \Delta y &= (x + \Delta x)^2 \\ &= (10 + 2)^2 \\ &= 144 \end{aligned}$$

As x increases from 10 to 12, y increases from 100 to 144 so that

$$\Delta x = 2$$

$$\Delta y = 44$$

and

$$\frac{\Delta y}{\Delta x} = \frac{44}{2} = 22$$

We are interested in the ratio $\frac{\Delta y}{\Delta x}$ because the limit of this ratio as Δx approaches zero is the derivative of

$$y = f(x)$$

As you recall from the discussion of limits, as Δx is made smaller, Δy gets smaller also. For our problem, the ratio $\frac{\Delta y}{\Delta x}$ approaches 20. This is shown in table 4-1.

Table 4-1.—Slope Values

Variable	Values of the variable						
Δx	2	1	0.5	0.2	0.1	0.01	0.0001
Δy	44	21	10.25	4.04	2.01	0.2001	0.00200001
$\frac{\Delta y}{\Delta x}$	22	21	20.5	20.2	20.1	20.01	20.0001

We may use a much simpler way to find that the limit of $\frac{\Delta y}{\Delta x}$ as Δx approaches zero is, in this case, equal to 20. We have two equations

$$y + \Delta y = (x + \Delta x)^2$$

and

$$y = x^2$$

By expanding the first equation so that

$$y + \Delta y = x^2 + 2x\Delta x + (\Delta x)^2$$

and subtracting the second from this, we have

$$\Delta y = 2x\Delta x + (\Delta x)^2$$

Dividing both sides of the equation by Δx gives

$$\frac{\Delta y}{\Delta x} = 2x + \Delta x$$

Now, taking the limit as Δx approaches zero, gives

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 2x$$

Thus,

$$\frac{dy}{dx} = 2x \tag{1}$$

NOTE: Equation (1) is one way of expressing the derivative of y with respect to x. Other ways are

$$\frac{dy}{dx} = y' = f'(x) = D(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

Equation (1) has the advantage that it is exact and true for all values of x. Thus if

$$x = 10$$

then

$$\frac{dy}{dx} = 2(10) = 20$$

and if

$$x = 3$$

then

$$\frac{dy}{dx} = 2(3) = 6$$

This method for obtaining the derivative of y with respect to x is general and may be formulated as follows:

1. *Set up the function of x as a function of $(x + \Delta x)$ and expand this function.*

2. *Subtract the original function of x from the new function of $(x + \Delta x)$.*

3. *Divide both sides of the equation by Δx.*

4. *Take the limit of all the terms in the equation as Δx approaches zero. The resulting equation is the derivative of $f(x)$ with respect to x.*

GENERAL FORMULA

To obtain a formula for the derivative of any expression in x, assume the function

$$y = f(x) \tag{4.2}$$

so that

$$y + \Delta y = f(x + \Delta x) \tag{4.3}$$

Subtracting equation (4.2) from equation (4.3) gives

$$\Delta y = f(x + \Delta x) - f(x)$$

and dividing both sides of the equation by Δx, we have

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

The desired formula is obtained by taking the limit of both sides as Δx approaches zero so that

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

or

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

NOTE: The notation $\frac{dy}{dx}$ is not to be considered as a fraction in which dy is the numerator and dx is the denominator. The expression $\frac{\Delta y}{\Delta x}$ is a fraction with Δy as its numerator and Δx as its denominator. Whereas, $\frac{dy}{dx}$ is a symbol representing the limit approached by $\frac{\Delta y}{\Delta x}$ as Δx approaches zero.

EXAMPLES OF DIFFERENTIATION

In this last section of the chapter, we will use several examples of differentiation to obtain a firm understanding of the general formula.

EXAMPLE: Find the derivative, $\frac{dy}{dx}$, for the function

$$y = 5x^3 - 3x + 2$$

determine the slopes of the tangent lines to the curve at

$$x = -1, \frac{-1}{\sqrt{5}}, 0, \frac{1}{\sqrt{5}}, 1$$

and draw the graph of the function.

SOLUTION: Finding the derivative by formula, we have

$$f(x + \Delta x)$$

$$= 5(x + \Delta x)^3 - 3(x + \Delta x) + 2 \qquad (1)$$

and

$$f(x) = 5x^3 - 3x + 2 \qquad (2)$$

Expand equation (1), then subtract equation (2) from equation (1), and simplify to obtain

$$f(x + \Delta x) - f(x)$$

$$= 5[3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3] - 3\Delta x$$

Dividing both sides by Δx, we have

$$\frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= 5[3x^2 + 3x\Delta x + (\Delta x)^2] - 3$$

Take the limit of both sides as $\Delta x \to 0$:

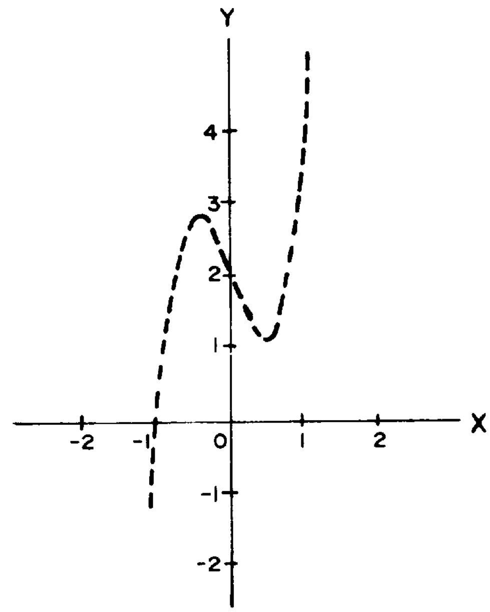
$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = 15x^2 - 3$$

Then

$$\frac{dy}{dx} = 15x^2 - 3 \qquad (3)$$

The slopes of the tangent lines to the curve at the points given, using this derivative, are shown in figure 4-7, view B.

Thus we have a new method of graphing an equation. By substituting different values of x in equation (3), we can find the slope of the tangent line to the curve at the point corresponding to the value of x. The graph of the curve is shown in figure 4-7, view A.



(A)

x	-1	$-\frac{1}{\sqrt{5}}$	0	$\frac{1}{\sqrt{5}}$	1
$\frac{dy}{dx}$	12	0	-3	0	12
y	0	2.89	2	1.1	4

(B)

Figure 4-7.—(A) Graph of $f(x) = 5x^3 - 3x + 2$; (B) chart of values.

EXAMPLE: Differentiate the function; that is, find $\frac{dy}{dx}$ of

$$y = \frac{1}{x}$$

and then find the slope of the tangent line to the curve at

$$x = 2$$

SOLUTION: Apply the formula for the derivative, and simplify as follows:

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{\frac{1}{x + \Delta x} - \frac{1}{x}}{\Delta x}$$

$$= \frac{\frac{x - (x + \Delta x)}{x(x + \Delta x)}}{\Delta x}$$

$$= \frac{-1}{x(x + \Delta x)}$$

Now take the limit of both sides as $\Delta x \to 0$ so that

$$\frac{dy}{dx} = \frac{-1}{x^2}$$

To find the slope of the tangent line to the curve at the point where x has the value 2, substitute 2 for x in the expression for $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{-1}{2^2}$$

$$= \frac{-1}{4}$$

EXAMPLE: Find the slope of the tangent line to the curve

$$f(x) = x^2 + 4$$

at

$$x = 3$$

SOLUTION: We need to find $\frac{dy}{dx}$, which is the slope of the tangent line at a given point. Apply the formula for the derivative as follows:

$$f(x + \Delta x) = (x + \Delta x)^2 + 4 \tag{1}$$

and

$$f(x) = x^2 + 4 \tag{2}$$

Expand equation (1) so that

$$f(x + \Delta x) = x^2 + 2x\Delta x + (\Delta x)^2 + 4$$

Then subtract equation (2) from equation (1):

$$f(x + \Delta x) - f(x) = 2x\Delta x + (\Delta x)^2$$

Now, divide both sides by Δx:

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} = 2x + \Delta x$$

Then take the limit of both sides as $\Delta x \rightarrow 0$:

$$\frac{dy}{dx} = 2x$$

Substitute 3 for x in the expression for the derivative to find the slope of the tangent line at

$$x = 3$$

so that

$$\text{slope} = 6$$

In this last example we will set the derivative of the function, $f(x)$, equal to zero and determine the values of the independent variable that will make the derivative equal to zero to find a *maximum* or *minimum point on the curve*. By maximum or minimum of a curve, we mean the point or points through which the slope of the tangent line to the curve changes from positive to negative or from negative to positive.

NOTE: *When the derivative of a function is set equal to zero, that does not mean in all cases we will have found a maximum or minimum point on the curve.* A complete discussion of maxima or minima may be found in most calculus texts.

To set the derivative equal to zero, we will require that the following conditions be met:

1. We have a maximum or minimum point.

2. The derivative exists.

3. We are dealing with an interior point on the curve.

When these conditions are met, the derivative of the function will be equal to zero.

EXAMPLE: Find the derivative of the function

$$y = 5x^3 - 6x^2 - 3x + 3$$

set the derivative equal to zero, and find the points of maximum and minimum on the curve. Then verify this by drawing the graph of the curve.

SOLUTION: Apply the formula for $\frac{dy}{dx}$ as follows:

$$f(x + \Delta x) = 5(x + \Delta x)^3 - 6(x + \Delta x)^2 - 3(x + \Delta x) + 3 \qquad (1)$$

and

$$f(x) = 5x^3 - 6x^2 - 3x + 3 \qquad (2)$$

Expand equation (1) and subtract equation (2), obtaining

$$f(x + \Delta x) - f(x) =$$

$$5(3x^2\Delta x + 3x\Delta x^2 + \Delta x^3) - 6(2x\Delta x + \Delta x^2) - 3\Delta x$$

Now, divide both sides by Δx, and take the limit as $\Delta x \to 0$ so that

$$\frac{dy}{dx} = 5(3x^2) - 6(2x) - 3$$

$$= 15x^2 - 12x - 3$$

Set $\frac{dy}{dx}$ equal to zero; thus

$$15x^2 - 12x - 3 = 0$$

Then

$$3(5x^2 - 4x - 1) = 0$$

and

$$(5x + 1)(x - 1) = 0$$

Set each factor equal to zero and find the points of maximum or minimum; that is,

$$5x = -1$$

$$x = \frac{-1}{5}$$

and

$$x = 1$$

The graph of the function is shown in figure 4-8.

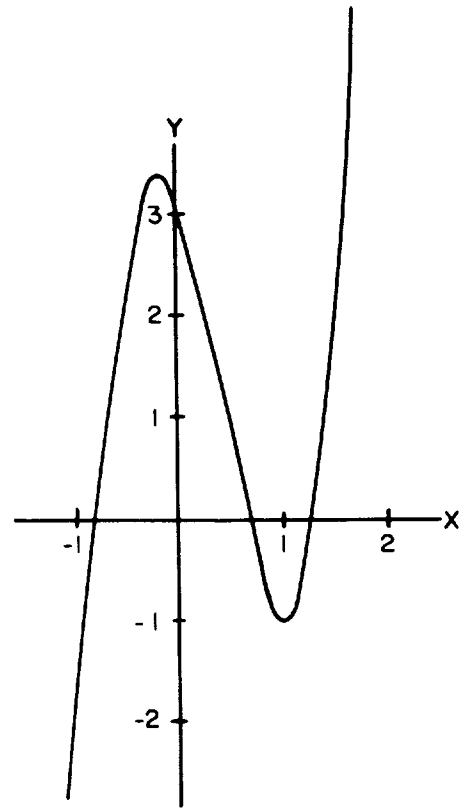


Figure 4-8.—Graph of $f(x) = 5x^3 - 6x^2 - 3x + 3$.

PRACTICE PROBLEMS:

Differentiate the functions in problems 1 through 3.

1. $f(x) = x^2 - 3$

2. $f(x) = x^2 - 5x$

3. $f(x) = 3x^2 - 2x + 3$

4. Find the slope of the tangent line to the curve

$$y = x^3 - 3x + 2$$

at the points

$$x = -2, 0, \text{ and } 3$$

5. Find the values of x where the function

$$f(x) = 2x^3 - 9x^2 - 60x + 12$$

has a maximum or a minimum.

ANSWERS:

1. $2x$
2. $2x - 5$
3. $6x - 2$
4. $m = 9, -3$, and 24
5. $x = -2$, $x = 5$

SUMMARY

The following are the major topics covered in this chapter:

1. **Definition of a limit**: Let $f(x)$ be defined for all x in the interval near $x = a$. Then there exists a number, L, such that for every positive number ε, however small, $| f(x) - L | < \varepsilon$, provided that we may find a positive number δ such that $0 < | x - a | < \delta$. Then we say L is the limit of $f(x)$ as x approaches a, and we write $\lim_{x \to a} f(x) = L$. This means that for every given number $\varepsilon > 0$, we must find a number δ such that the difference between $f(x)$ and L is smaller than the number ε whenever $0 < | x - a | < \delta$.

2. **Indeterminate forms:**

$$0/0, \infty/\infty, \ (\infty)0, \ 0^0, \infty^0, \ 1^\infty$$

   Two methods of evaluating indeterminate forms are (1) factoring and (2) division of the numerator and denominator by powers of the variable.

3. **Limit theorems:**

   Theorem 1. *The limit of the sum of two functions is equal to the sum of the limits*:

$$\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

   This theorem can be extended to include any number of functions, such as

$$\lim_{x \to a} [f(x) + g(x) + h(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) + \lim_{x \to a} h(x)$$

   Theorem 2. *The limit of a constant, c, times a function, f(x), is equal to the constant, c, times the limit of the function:*

$$\lim_{x \to a} cf(x) = c \lim_{x \to a} f(x)$$

   Theorem 3. *The limit of the product of two functions is equal to the product of their limits:*

$$\lim_{x \to a} f(x)\, g(x) = \left[\lim_{x \to a} f(x)\right] \left[\lim_{x \to a} g(x)\right]$$

Theorem 4. *The limit of the quotient of two functions is equal to the quotient of their limits, provided the limit of the divisor is not equal to zero:*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}, \text{ if } \lim_{x\to a} g(x) \neq 0$$

4. **Infinitesimals**: A variable that approaches 0 as a limit is called an *infinitesimal*:

$$\lim V = 0 \text{ or } V \to 0$$

The difference between a variable and its limit is an infinitesimal:

$$\text{If } \lim V = L, \text{ then } \lim V - L = 0$$

5. **Sum and product of infinitesimals:**

Theorem 1. *The algebraic sum of any number of infinitesimals is an infinitesimal.*

Theorem 2. *The product of any number of infinitesimals is an infinitesimal.*

Theorem 3. *The product of a constant and an infinitesimal is an infinitesimal.*

6. **Continuity**: A function, $f(x)$, is *continuous* at $x = a$ if the following three conditions are met:

   1. $f(x)$ is defined at $x = a$.
   2. The limit of $f(x)$ exists as x approaches a or $x \to a$.
   3. The value of $f(x)$ at $x = a$ is equal to the limit of $f(x)$ at $x = a$ or $\lim\limits_{x\to a} f(x) = f(a)$.

7. **Discontinuity**: If a function is not continuous at $x = a$, then it is said to be *discontinuous* at $x = a$.

8. **Ways of expressing the derivative of y with respect to x:**

$$\frac{dy}{dx} = y' = f'(x) = D(x) = \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x}$$

9. **Increment method for obtaining the derivative of y with respect to x:**

   1. Set up the function of x as a function of $(x + \Delta x)$ and expand this function.

   2. Subtract the original function of x from the new function of $(x + \Delta x)$.

   3. Divide both sides of the equation by Δx.

   4. Take the limit of all the terms in the equation as Δx approaches zero. The resulting equation is the derivative of $f(x)$ with respect to x.

10. **General formula for the derivative of any expression in x:**

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

11. **Maximum or minimum points on a curve:** Set the derivative of the function, $f(x)$, equal to zero and determine the values of the independent variable that will make the derivative equal to zero. (Note: When the derivative of a function is set equal to zero, that does not mean in all cases the curve will have a maximum or minimum point.)

ADDITIONAL PRACTICE PROBLEMS

Find the limit of each of the following:

1. $\lim_{x\to 0} \sqrt{\dfrac{x+16}{3x+4}}$

2. $\lim_{t\to -2} \dfrac{3t^2+t}{t-3}$

3. $\lim_{x\to -3} \dfrac{x^2-9}{x^2+5x+6}$

4. $\lim_{h\to 0} \dfrac{(2+h)^2-4}{h}$

5. $\lim_{x\to \infty} \dfrac{3+2x+10x^2}{2x^2+8}$

6. $\lim_{x\to 9} \sqrt{x}+(x-6)^2$

   (using Limit Theorem 1.)

7. $\lim_{x\to 4} 6\sqrt{(x+5)}$

   (using Limit Theorem 2.)

8. $\lim_{x\to 3} \left(\dfrac{15}{x+2}\right)\left(\dfrac{8x}{x^2-3}\right)$

   (using Limit Theorem 3.)

9. $\lim_{x\to 5} \dfrac{60/(1+x)+2}{16/(x^2-21)}$

   (using Limit Theorem 4.)

10. Find where

$$f(x) = \frac{(x^2+6x+8)(x-3)}{x^2-x-6}$$

is discontinuous and then extend the equation so that the function is continuous.

11. Differentiate $f(x) = 6/x - 1$.

12. Find the slope of the tangent line to the curve $y = 3x^2 - 9x$ at the points $x = 0$ and 3.

13. Find the values of x where the function $f(x) = x(4 - 2x)$ has a maximum or a minimum.

ANSWERS TO ADDITIONAL PRACTICE PROBLEMS

1. 2
2. -2
3. 6
4. 4
5. 5
6. 12
7. 18
8. 12
9. 3
10. $x = -2, 3$

    $f(-2) = 2$

    $f(3) = 7$
11. $-6/x^2$
12. $m = -9$ and 9
13. $x = 1$